GENERALIZED METHOD OF CELLS

In the original formulation of the method of cells, a continuously (or discontinuously) reinforced, unidirectional fibrous composite is modeled as a rectangular, double-periodic (or triply-periodic) array of fibers embedded in a matrix phase. The periodic character of the assemblage allows one to identify a repeating unit cell that can be used as a building block to construct the entire composite. The properties of the repeating cell are thus representative of the properties of the entire assemblage. The unit cell consists of a single fiber subcell surrounded by three matrix subcells for continuous and seven for discontinuous composites, hence the name **method of cells**. The rectangular geometry of the repeating unit cell allows one to obtain an approximate solution for the stresses and strains in the individual subcells given some macroscopically homogeneous state of strain or stress applied to the composite. The approximate solution to the posed boundary value problem is, in turn, used to determine macroscopic (average) or effective properties of the composite and the effective stress-strain response in the inelastic region.

In the **generalized method of cells** for continuous (or discontinuous) fibrous composites, the repeating unit cell can consist of an arbitrary number of phases. Hence the generalized method of cells is capable of modeling a multiphase composite. This generalization extends the modeling capability of the original method of cells to include the following: 1) inelastic thermomechanical response of multiphased metal matrix composite, 2) modeling of various fiber architectures (including both shape and packing arrangements), 3) modeling of porosities and damage, and 4) the modeling of interfacial regions around inclusions including interfacial degradation.

The basic homogenization approach taken in the micromechanical analysis consists essentially of four steps. First, the repeating volume element, RVE, of the periodic composite is identified. Second, the macroscopic or average stress and strain state in terms of the individual microscopic (subcell) stress and strain states is defined. Third, the continuity of tractions and displacements are imposed at the boundaries between the constituents. These three steps, in conjunction with micro-equilibrium, establish the relationship between micro (subcell) total, thermal and inelastic strains and macro (composite) strains via the relevant concentration tensors. In the fourth and final step, the overall macro constitutive equations of the composite are determined. These four steps form the basis of the micro-to-macromechanics analysis which describe the behavior of heterogeneous media. The resulting micromechanical analysis establishes the overall (macro) behavior of the multi-phase composite and is expressed as a constitutive relation between the average stress, strain, thermal, and inelastic strains, in conjunction with the effective elastic stiffness tensor.

That is,

$$\bar{\underset{\sim}{\sigma}} = \underset{\sim}{B}^{*}(\bar{\underset{\sim}{\varepsilon}} - \bar{\underset{\sim}{\varepsilon}}^{I} - \bar{\underset{\sim}{\varepsilon}}^{T}) \qquad \text{(EQ 1)}$$

where for the most general case of discontinuous reinforcement with N_{α} by N_{β} by N_{γ} number of subcells, the effective elastic stiffness tensor, $\underset{\sim}{B}^{*}$, of the composite is given by,

$$\underset{\sim}{B}^{*} = \frac{1}{dhl}\sum_{\alpha=1}^{N_\alpha}\sum_{\beta=1}^{N_\beta}\sum_{\gamma=1}^{N_\gamma} d_\alpha h_\beta l_\gamma \underset{\sim}{C}^{(\alpha\beta\gamma)}\underset{\sim}{A}^{(\alpha\beta\gamma)} \tag{EQ 2}$$

the composite inelastic strain tensor is defined as,

$$\bar{\underset{\sim}{\varepsilon}}^{I} = \frac{-\underset{\sim}{B}^{*-1}}{dhl}\sum_{\alpha=1}^{N_\alpha}\sum_{\beta=1}^{N_\beta}\sum_{\gamma=1}^{N_\gamma} d_\alpha h_\beta l_\gamma \underset{\sim}{C}^{(\alpha\beta\gamma)}\left(\underset{\sim}{D}^{(\alpha\beta\gamma)}\underset{\sim}{\varepsilon}_s^{I} - \bar{\underset{\sim}{\varepsilon}}^{I(\alpha\beta\gamma)}\right) \tag{EQ 3}$$

the average thermal strain tensor as,

$$\bar{\underset{\sim}{\varepsilon}}^{T} = \frac{-\underset{\sim}{B}^{*-1}}{dhl}\sum_{\alpha=1}^{N_\alpha}\sum_{\beta=1}^{N_\beta}\sum_{\gamma=1}^{N_\gamma} d_\alpha h_\beta l_\gamma \underset{\sim}{C}^{(\alpha\beta\gamma)}\left(\underset{\sim}{D}^{(\alpha\beta\gamma)}\underset{\sim}{\varepsilon}_s^{T} - \bar{\underset{\sim}{\varepsilon}}^{T(\alpha\beta\gamma)}\right) \tag{EQ 4}$$

and $\bar{\underset{\sim}{\varepsilon}}$ is the uniform applied macro (composite) strain. For the case of continuous reinforcements with N_β by N_γ number of subcells, eq. (2) - (4) reduce to the following:

$$\underset{\sim}{B}^{*} = \frac{1}{hl}\sum_{\beta=1}^{N_\beta}\sum_{\gamma=1}^{N_\gamma} h_\beta l_\gamma \underset{\sim}{C}^{(\beta\gamma)}\underset{\sim}{A}^{(\beta\gamma)} \tag{EQ 5}$$

$$\bar{\underset{\sim}{\varepsilon}}^{I} = \frac{-\underset{\sim}{B}^{*-1}}{hl}\sum_{\beta=1}^{N_\beta}\sum_{\gamma=1}^{N_\gamma} h_\beta l_\gamma \underset{\sim}{C}^{(\beta\gamma)}\left(\underset{\sim}{D}^{(\beta\gamma)}\underset{\sim}{\varepsilon}_s^{I} - \bar{\underset{\sim}{\varepsilon}}^{I(\beta\gamma)}\right) \tag{EQ 6}$$

$$\bar{\underset{\sim}{\varepsilon}}^{T} = \frac{-\underset{\sim}{B}^{*-1}}{hl}\sum_{\beta=1}^{N_\beta}\sum_{\gamma=1}^{N_\gamma} h_\beta l_\gamma \underset{\sim}{C}^{(\beta\gamma)}\left(\underset{\sim}{D}^{(\beta\gamma)}\underset{\sim}{\varepsilon}_s^{T} - \bar{\underset{\sim}{\varepsilon}}^{T(\beta\gamma)}\right) \tag{EQ 7}$$

In the above equations matrix notation is employed; where, for example, the average stress, $\bar{\underset{\sim}{\sigma}}$, average applied strain, $\bar{\underset{\sim}{\varepsilon}}$, and inelastic subcell strain, $\underset{\sim}{\varepsilon}_s^{I}$, vectors represent,

$$\bar{\underset{\sim}{\sigma}} = \{\bar{\sigma}_{11}, \bar{\sigma}_{22}, \bar{\sigma}_{33}, \bar{\sigma}_{12}, \bar{\sigma}_{23}, \bar{\sigma}_{13}\} \tag{EQ 8}$$

$$\bar{\underset{\sim}{\varepsilon}} = \{\bar{\varepsilon}_{11}, \bar{\varepsilon}_{22}, \bar{\varepsilon}_{33}, \bar{\varepsilon}_{12}, \bar{\varepsilon}_{23}, \bar{\varepsilon}_{13}\} \tag{EQ 9}$$

$$\underset{\sim}{\varepsilon}_s^{I} = \{\bar{\underset{\sim}{\varepsilon}}^{I(111)}, \ldots, \bar{\underset{\sim}{\varepsilon}}^{I(N_\alpha N_\beta N_\gamma)}\} \tag{EQ 10}$$

where the six components of the vector $\bar{\underset{\sim}{\varepsilon}}^{I(\alpha\beta\gamma)}$ are arranged as in eq. (9). Similar definitions for $\underset{\sim}{\varepsilon}_s^{T}$, $\bar{\underset{\sim}{\varepsilon}}^{T(\alpha\beta\gamma)}$ also exist. Note that the key ingredient in the construction of this macro constitutive law is the derivation of the appropriate concentration matrices, $\underset{\sim}{A}^{(\alpha\beta\gamma)}$ and $\underset{\sim}{D}^{(\alpha\beta\gamma)}$ having the dimensions 6 by 6 and 6 by $N_\alpha N_\beta N_\gamma$ respectively, at the micro (subcell) level. The definitions of $\underset{\sim}{A}$ and $\underset{\sim}{D}$, although not given here, may be found in references [2] and [5]. Finally, the matrix $\underset{\sim}{C}^{(\alpha\beta\gamma)}$ represents the elastic stiffness tensor of each subcell $(\alpha\beta\gamma)$ and $d_\alpha, h_\beta, l_\gamma$ the respective subcell dimensions wherein,

$$d = \sum_{\alpha = 1}^{N_\alpha} d_\alpha \qquad h = \sum_{\beta = 1}^{N_\beta} h_\beta \qquad l = \sum_{\gamma = 1}^{N_\gamma} l_\gamma$$

Similarly, given the concentration matrices $\underset{\sim}{A}^{(\alpha\beta\gamma)}$ and $\underset{\sim}{D}^{(\alpha\beta\gamma)}$, expressions for the average strain in each subcell can be constructed, i. e.,

$$\bar{\underset{\sim}{\varepsilon}}^{(\alpha\beta\gamma)} = \underset{\sim}{A}^{(\alpha\beta\gamma)}\bar{\underset{\sim}{\varepsilon}} + \underset{\sim}{D}^{(\alpha\beta\gamma)}(\underset{\sim}{\varepsilon}_s^I + \underset{\sim}{\varepsilon}_s^T)$$

as well as average stress,

$$\bar{\underset{\sim}{\sigma}}^{(\alpha\beta\gamma)} = \underset{\sim}{C}^{(\alpha\beta\gamma)}[\underset{\sim}{A}^{(\alpha\beta\gamma)}\bar{\underset{\sim}{\varepsilon}} + \underset{\sim}{D}^{(\alpha\beta\gamma)}(\underset{\sim}{\varepsilon}_s^I + \underset{\sim}{\varepsilon}_s^T) - (\underset{\sim}{\varepsilon}^{I(\alpha\beta\gamma)} + \underset{\sim}{\varepsilon}^{T(\alpha\beta\gamma)})]$$

The analytic constitutive law, see eq. 1, may be readily applied to investigate the behavior of various types of composites, given knowledge of the behavior of the individual phases. Numerous advantages can be stated regarding the current macro/micro constitutive laws as compared to the other numerical micromechanical approaches in the literature, e.g. the finite element unit cell approach. One advantage is that any type of simple or combined loading (multiaxial state of stress) can be applied irrespective of whether symmetry exists or not, as well as without resorting to different boundary condition application strategies as in the case of the finite element unit cell procedure. Another, advantage concerns the availability of an analytical expression representing the macro elastic-thermo-inelastic constitutive law thus ensuring a reduction in memory requirements when implementing this formulation into a structural finite element analysis code. Furthermore, this formulation has been shown to predict accurate macro behavior given only a few subcells, within the repeating cell (see references [2], and [4]). Whereas, if one employs the finite element unit cell procedure, a significant number of finite elements are required within a given repeating unit cell to obtain the same level of accuracy as with the present formulation. Consequently, it is possible to utilize this formulation to efficiently analyze metal matrix composite structures subjected to complex thermomechanical load histories. This is particularly important when analyzing realistic structural components, since different loading conditions exist throughout the structure, thus necessitating the application of the macromechanical equations repeatedly at these locations.